RATIONAL APPROXIMATIONS WITH BOUNDED DENOMINATORS

Mark D. Bratcher
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Introduction

When performing fractional arithmetic, there is typically a limit on the allowable magnitude of the denomintor either established arbitrarily or because of limits in the representation of integers on the target platform. Operations on the fractions, therefore, quickly lead you to the following problem.

> *Given a real number, x, and a fixed positive integer, N, find the rational fraction a/b, with* $b \leq N$*, which best approximates x.*

A *rational fraction* is simply a fraction with integers for its numerator and denominator. A rational fraction approximation to a real number, per the conditions given above, is simple to compute using a brute force method shown in *C* in Figure 1. The performance of this computation, however, is $O(n)$. Therefore, it begs a more efficient and elegant approach (or, dare we say, a more *rational* approach).

```c
int a_save, b_save, a, b;
double x;

a_save = b_save = 1;

/* Assumes 0 <= x <= 1 */

/* The following loop computes the best rational
   approximation for x with denominator <= N */

for ( a = x, b = 1; b <= N; b++, a += x ) {
     /* compute best numerator for denominator b */

     if ( abs(x - (double)a/b) <
               abs(x - (double)a_save/b_save) ) {
          a_save = a;
          b_save = b;
     }
}

/* a_save/b_save is the desired approximation */
```

Figure 1. *Brute force bounded rational approximation*

Farey Sequences

The topic of rational approximations to real numbers has elaborate coverage in the field of Number Theory. One study in particular involves a very interesting classification of sequential rational fractions known as a *Farey sequence* which is defined as follows.

> *The* Farey sequence of order n *is the sequence of all reduced (lowest terms) fractions with denominators not exceeding* n *and listed in order of their magnitude. Fractions that are being classified as part of a given Farey sequence are called* Farey fractions.

Let's limit our discussion to numbers that fall between 0 and 1. (It is simple to extend it to values outside this range.) Figure 2 shows how to enumerate the Farey sequences up to order 5 for rational numbers between 0 and 1. In general, start with the first row consisting of the fractions 0/1 and 1/1 at each end (order 1 Farey sequence). Then construct the nth row (n = 2, 3, ...) by copying the previous row in order and inserting the fraction $(x + x')/(y + y')$ between the consecutive fractions x/y and x'/y' from the previous row only if $y + y'$ does not exceed n. It can easily be proven that the nth row of this construction is indeed the Farey sequence of order n.[1]

0/1										1/1
0/1					1/2					1/1
0/1			1/3		1/2		2/3			1/1
0/1		1/4	1/3		1/2		2/3	3/4		1/1
0/1	1/5	1/4	1/3	2/5	1/2	3/5	2/3	3/4	4/5	1/1

Figure 2. *Building the Farey sequences of orders 1 through 5.*

Some important provable[2] facts pertaining to this construction are:

- If x/y and x'/y' are non-negative rational fractions and $x/y \le x'/y'$, then $x/y \le (x + x')/(y + y') \le x'/y'$.

- If x/y and x'/y' are consecutive fractions in a given Farey sequence, then $(x + x')/(y + y')$ is in lowest terms.

Application

Now this gives us an almost binary search approach to zooming in on a rational approximation to a given real number. Figure 3 shows an example algorithm based upon the Farey sequence construction concept. Rather than being strictly $O(n)$, this new algorithm varies between $O(n)$ and $O(\log_2 n)$ in performance.

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[1]Niven & Zuckerman, *An Introduction to The Theory of Numbers*, Third Edition, pp. 134-7 (1972).
[2]*Ibid.*, p. 134.

Why does this work? How do you know that we don't miss any fractions, with denominators in range, that are close to x? The algorithm works because, as proven by the cited theorems, our construction generates Farey fractions, and a Farey sequence enumerates *all* fractions whose denominators do not exceed a given value.

There is yet another improvement that can be made. Note that the stopping criteria for the loop in Figure 3 is at the point of maximum denominator. This might be long after the best approximation has been obtained.

To remedy this, we can apply the following theorem[3] regarding the approximation of reals using Farey fractions.

> *If n is a positive integer and x is a real number, then there is a rational number a/b such that $0 < b \le n$ and*
>
> $$\left| x - \frac{a}{b} \right| \le \frac{1}{b(n+1)} .$$

Although this theorem does not establish uniqueness of a/b for a given n, it gives an excellent termination criterion for our loop. That is, we can test the absolute difference $(x - a/b)$ against $b \times (N + 1)$ and, if the former is less than the latter, exit the loop sooner.

[3] *Ibid.*, p. 138.

```
int a_save, b_save;
int a_mid, b_mid;
int a_left, b_left;
int a_right, b_right;
double x;

/* Initial interval check is 0/1 -> 1/1. Assumes 0 <= x <= 1. */

a_left = 0;
a_right = b_left = b_right = 1;

if ( 1 - x > x ) {
     a_save = a_left;
     b_save = b_left;
}
else {
     a_save = a_right;
     b_save = b_right;
}

/* This loop uses the Farey sequence to compute the best
   rational approximation to x with denominator <= N */

while (1) {
     /* Compute "middle" fraction. We're done if we exceed the
        maximum denominator. */

     b_mid = b_left + b_right;

     if ( b_new > N )
          break;

     a_mid = a_left + a_right;

     /* Is it left or right of x?
        Check abs(x - a_mid/b_mid) versus 0 */

     if ( (x * b_mid) < (double)a_mid ) {
          a_right = a_mid;    /* "mid" is our new "right" */
          b_right = b_mid;
     }
     else {
          a_left = a_mid;     /* "mid" is our new "left" */
          b_left = b_mid;
     }
```

Figure 3. *Bounded rational approximation using Farey fractions*

```
     /* is it the closest so far? */

     if ( abs(x - (double)a_new/b_new) <
               abs(x - (double)a_save/b_save) ) {
          a_save = a_new;
          b_save = b_new;
     }
}

/* a_save/b_save is the desired approximation */
```

Figure 3 (cont'd). *Bounded rational approximation using Farey fractions*